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Flux Equivalences among Rayleigh, Isotropic, and Other Scattering Models

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ABSTRACT

The mathematical treatment of multiple scattering processes leads to formidable functional equations. It is of great importance to find simple analytical expressions which provide useful approximations to the exact solutions. H. C. van de Hulst has done much along these lines.

This paper provides several such approximations for the cases of conservative isotropic and Rayleigh scattering in slabs. The approach is to find the transmitted and reflected fluxes for these cases for a wide range of slab thicknesses using high-precision numerical methods. Then simple analytical expressions are derived using idealized models. These are evaluated numerically, and the results are compared against the earlier calculations.

One of the interesting results is that, so far as reflected and transmitted fluxes are concerned, there is little difference between conservative isotropic scattering and Rayleigh scattering.

This paper raises an interesting mathematical question: What is the underlying reason for all of these flux equivalences, and how far does this equivalence extend to other situations?

1. INTRODUCTION

The reflection and transmission properties of media which absorb and multiple-scatter energy have been studied by many investigators [1]–[4] employing various analytical, numerical and approximate methods. Since even relatively simple physical models lead to complicated functional equations, it is clear that simple, approximate formulas are needed for many applications. Such formulas must first be tested for accuracy and range of validity against accurate numerical values [5]–[10] or exact analytical expressions.

In a recent paper, the remarkable accuracy of a simple formula for reflected flux from an isotropically scattering slab was established over virtually *all* optical thicknesses [7], [8].

In this paper, we compare fluxes of reflected and transmitted radiation for

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several scattering models. Two of the models—isotropic scattering in a slab, as treated by invariant embedding, and Rayleigh scattering in a slab—lead to sophisticated numerical calculations. Two other models yield simple formulas for emergent fluxes. Fluxes determined from all models are in remarkable agreement for all optical thicknesses from zero to fifty.

2. Slab with Isotropic Scattering

2.1. *Exact Equations for Reflection and Transmission*

Consider a horizontal, homogeneous, absorbing and isotropically scattering plane-parallel medium of finite optical thickness x. Monodirectional radiation with direction cosine u relative to the downward directed vertical is uniformly incident on the top surface of the slab. The case of normal incidence ($u = 1$) is of particular interest. The net flux per unit area normal to the incident rays is π. The albedo for single scattering is λ ($0 < \lambda \leqslant 1$). We shall here consider only the case of conservative scattering, $\lambda = 1$. The lower surface is a completely absorbing barrier.

Let $r(v, u, x)$ be the intensity of the multiply-scattered radiation which emerges from the top of the medium with direction cosine v relative to the upward directed vertical. Let $t(v, u, x)$ be the intensity of the diffusely transmitted radiation emerging from the bottom with direction cosine v relative to the donward vertical. The function $t(v, u, x)$ refers to radiation which has interacted one or more times with the medium. The intensity of the reduced incident radiation which is directly transmitted is $\frac{1}{2}\delta(u - v) \exp(-x/u)$, where $\delta(\ )$ denotes the Dirac delta function. This corresponds, of course, to a net reduced incident flux of $\pi u \exp(-x/u)$.

Reflected and transmitted fluxes computed via the invariant imbedding method (see Refs. [4], [7], [11], and [13]) are denoted $\rho_{\mathrm{I}}(u, x)$ and $\tau_{\mathrm{I}}(u, x)$.

For isotropic scattering, the reflected flux is defined to be

$$\rho(u, x) \equiv 2\pi \int_0^1 r(z, u, x) z\,dz, \tag{1}$$

the diffusely transmitted flux,

$$\tau(u, x) \equiv 2\pi \int_0^1 t(z, u, x) z\,dz, \tag{2}$$

and the total transmitted flux is

$$\tau(u, x) + \pi u e^{-x/u}. \tag{3}$$

The conservation law requires that

$$\pi u = \rho(u, x) + \tau(u, x) + \pi u e^{-x/u}. \tag{4}$$

This serves as a convenient check on numerical values of ρ and τ.

2.2. *Approximate Formulas of Gavallas and Kagan*

Gavallas and Kagan [8] have developed formulas for reflected and transmitted fluxes for the case of conservative scattering and normal incidence. They assumed that the intensities have the form

$$A + B\cos\theta,$$

where θ is the polar angle. Their final expressions for the fluxes, obtained with the use of the Bubnov–Galerkin method, are

$$\begin{aligned} \rho_G(1, x) &= \pi \frac{-1 + 3x + e^{-x}}{4 + 3x}, \\ \tau_G(1, x) &= \pi \frac{5 - 5e^{-x} - 3xe^{-x}}{4 + 3x}. \end{aligned} \tag{5}$$

3. Rayleigh Scattering in a Slab

Reflected and transmitted fluxes for Rayleigh slabs have been determined by Kahle, using a theory for singular integral equations. We refer the reader to her paper [12] and the references cited therein. The fluxes determined by Kahle are denoted $\rho_K(u, x)$ and $\tau_K(u, x)$. These fluxes could also have been obtained via an initial-value problem for Riccati equations [13].

4. One-Dimensional Model

Next we consider a simple model of isotropic scattering in a rod. Let a unit of energy per unit time be incident on the right end of a rod of length x. An analysis by invariant embedding leads to the equation for the reflected flux, $s(x)$,

$$s(x) = \frac{x}{x + 2}. \tag{6}$$

To make the proper comparisons with the other models, we set the incident flux equal to πu, and the effective length equal to x/u. Then the reflected flux is

$$\rho_0(u, x) = \pi u \frac{x/u}{(x/u) + 2}$$

or

$$\rho_0(u, x) = \pi u \frac{x}{x + 2u} . \tag{7}$$

From the conservation Eq. (4), the transmitted flux is seen to be

$$\tau_0(u, x) = \pi u \left[\frac{2u}{x + 2u} - e^{-x/u}\right]. \tag{8}$$

5. Numerical Results

Computations for the conservative case ($\lambda = 1$) in slab geometry are carried out with the invariant embedding equations for the slab. The order of the

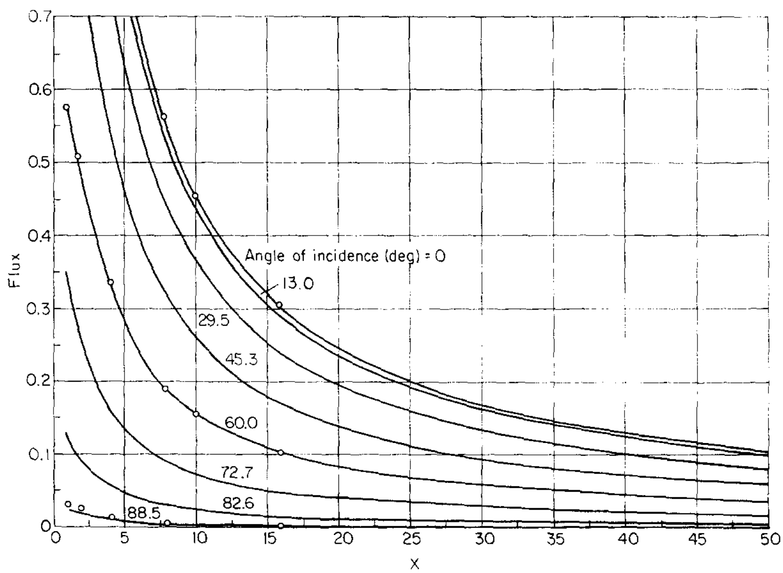

FIG. 1. Fluxes diffusely transmitted through conservatively scattering slabs with various angles of incidence. The curves are for the isotropic scattering case, the dots for Rayleigh scattering.

quadrature formula, N, and the step size of integration, Δx, are varied. The pairs of parameter values $(N, \Delta x)$ are (3, 0.05), (3, 0.01), (5, 0.01), (7, 0.01), and (7, 0.005). The effect of changing the step size from 0.01 to 0.005 when $N = 7$ is a change, at most, of one unit in the fifth significant figure. The same is true for $N = 3$ and the two step sizes tested. Both hold for thicknesses up to 50. The overall agreement among all of the trials is 3 to 4 decimal places. These calculations are performed on an IBM 7044 using a fourth-order Adams–Moulton integration routine.

The invariant embedding numerical results quoted below are obtained with $N = 7$ and $\Delta x = 0.01$. The computing time is 20 minutes for solving the system of 113 simultaneous ordinary differential equations on the interval $0 \leqslant x \leqslant 50$.

The formulas of Gavallas and Kagan are excellent approximations to the reflected and transmitted fluxes on the interval $0 \leqslant x \leqslant 50$. The greatest discrepancy from the invariant imbedding calculation is 0.0102 at $x = 2$. For the most part, the discrepancy is in the order of 0.001.

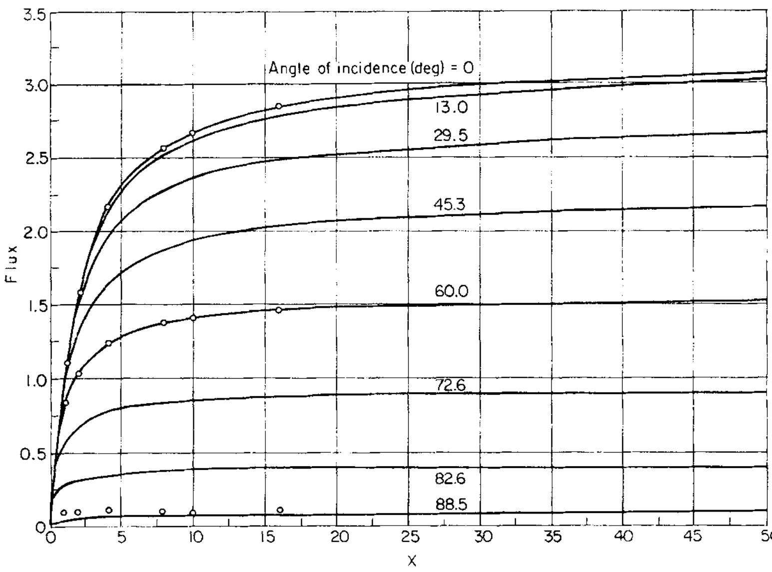

FIG. 2. Fluxes reflected from conservatively scattering slabs with various angles of incidence. The curves are for the isotropic scattering case, the dots for Rayleigh scattering.

Eight curves of diffusely transmitted flux are shown in Fig. 1 for the incident angles indicated. Note that the dots are plots of the values obtained by Kahle for Rayleigh scattering and incident angles 0, 60, and 88 (not 88.5) degrees. Kahle has also computed fluxes at $x = 100$. From this figure and by graphical interpolation, it is clear that the results coincide over all thicknesses up to 100. A similar conclusion can be drawn from Fig. 2 for reflected flux.

A brief tabular survey of the comparisons among the four methods of determining the reflected and diffusely transmitted fluxes is presented in Tables I and II.

TABLE I

A COMPARISON OF REFLECTED FLUXES

x	$\cos^{-1}u$ (deg)	ρ_0	ρ_I	ρ_G	ρ_K
0.15	0	0.2193	0.2198	0.2194	0.2196
	60	0.2048	0.2054	—	0.2054
	88.5	0.0596	0.0446	—	0.0598[a]
1.0	0	1.0471	1.0723	1.0627	1.0687
	60	0.7854	0.7829	—	0.7842
	88.5	0.0760	0.0591	—	0.0813[a]
10.0	0	2.6179	2.6800	2.6796	2.6762
	60	1.4279	1.4110	—	1.4120
	88.5	0.0795	0.0756	—	0.1038[a]
50.0	0	3.0206	3.0400	3.0396	—
	60	1.5400	1.5357	—	—
	88.5	0.0798	0.0790	—	—

[a] $\cos^{-1}u = 88.0$ degrees.

TABLE II

A COMPARISON OF DIFFUSELY TRANSMITTED FLUXES

x	$\cos^{-1}u$ (deg)	τ_0	τ_I	τ_G	τ_K
0.15	0	0.2184	0.2178	0.2182	0.2179
	60	0.2022	0.2018	—	0.2017
	88.5	0.0200	0.0351	—	0.0484[a]
1.0	0	0.9387	0.9136	0.9232	0.9172
	60	0.5728	0.5754	—	0.5740
	88.5	0.0039	0.0208	—	0.0284[a]
10.0	0	0.5235	0.4618	0.4619	0.4653
	60	0.1428	0.1599	—	0.1588
	88.5	0.0004	0.0044	—	0.0059[a]
50	0	0.1208	0.1031	0.1020	—
	60	0.0308	0.0357	—	—
	88.5	0.0001	0.0010	—	—

[a] $\cos^{-1}u = 88.0$ degrees.

6. Discussion

Let us summarize the results of this paper. Precision calculations for isotropic scattering and Rayleigh scattering in slabs have been performed. They have served to establish that certain simple formulas provide excellent approximations to the diffusely reflected and transmitted fluxes. Furthermore, they have shown that the external fluxes are virtually independent of the nature of the local scattering law, for the cases considered.

Still lacking are physical and mathematical explanations of these flux equivalences. Note that, in the cases considered, the scattering diagrams were the same in the forward hemisphere as in the backward. It has been suggested that this symmetry must be an important factor [14].

The conservative nature of the scattering means that none of the energy is truly absorbed but that the energy is ultimately reflected or transmitted. The ratio of reflected to transmitted flux has here been shown to be dependent on the optical thickness and the direction of illumination, but very insensitive to the phase function.

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